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The behavior of the solutions of periodic linear neutral delay difference equations

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Abstract

Some new asymptotic, nonoscillation and stability criteria for linear neutral delay difference equations with periodic coefficients and constant delays are given. The results are obtained via a positive root (with suitable properties) of an associated equation which is, in a sense, the corresponding characteristic equation.

MSC: 39A10

Keywords: Difference equation; Delay; Periodic coefficients; Solution; Asymptotic behavior; Nonoscillation; Stability

1. Introduction

This paper deals with the asymptotic behavior, the nonoscillation and the stability of the solutions of linear neutral delay difference equations with periodic coefficients and constant delays, where the coefficients have a common period and the delays are multiples of this period. The results obtained include (as a special case) those due to Kordonis et al. [14], which concern linear (nonneutral) delay difference equations with periodic coefficients having a common period and constant delays that are multiples of this period. Also, the results due to Kordonis and Philos [12] for linear autonomous neutral delay difference equations are included (as a special case) in the results given here. Our results should be looked upon as the discrete analogues of the ones obtained by Philos and Purnaras [21] on the asymptotic behavior, the nonoscillation and the stability of the solutions of first order linear neutral delay differential

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equations in which the coefficients are periodic with the same period and the delays are constant and multiples of the period. It should be noted that the treatment of the problem studied in this paper for difference equations displays more difficulties and complexities than the treatment of the corresponding problem in the case of differential equations. For the basic theory of difference equations, we choose to refer to the books [1,6,9,15,16].

Some related asymptotic results for linear delay difference equations can be found in [2,4,8,17,22,23]. Also, we notice that linear delay difference equations with periodic coefficients have been studied (with respect to oscillation) in [13,18,20]; for some oscillation results for linear neutral delay difference equations with periodic coefficients, we choose to refer to [11]. Moreover, we note that the results in [21] extend and unify the ones given by Philos [19] and Kordonis et al. [10] (and, in particular, slightly improve those in [19]); for previous related results, see [3,5,7].

Consider the neutral delay difference equation

$$\Delta\left(x_n+\sum_{i\in I}c_i x_{n-\sigma_i}\right)=a(n)x_n+\sum_{j\in J}b_j(n)x_{n-\tau_j}, \tag{E}$$

where *I and J are initial segments of natural numbers, c_i for $i\in I$ are real numbers, $(a(n))_{n\geqslant 0}$ and $(b_j(n))_{n\geqslant 0}$ for $j\in J$ are sequences of real numbers, σ_i for $i\in I$ are positive integers such that $\sigma_{i_1}\neq\sigma_{i_2}$ for $i_1, i_2\in I$ with $i_1\neq i_2$, and τ_j for $j\in J$ are positive integers such that $\tau_{j_1}\neq\tau_{j_2}$ for $j_1, j_2\in J$ with $j_1\neq j_2$. It is assumed that the sequences $(b_j(n))_{n\geqslant 0}$ for $j\in J$ are not identically zero. Moreover, it will be supposed that the coefficients $(a(n))_{n\geqslant 0}$ and $(b_j(n))_{n\geqslant 0}$ for $j\in J$ are periodic sequences with a common period T (where T is a positive integer) and that there exist positive integers ℓ_i for $i\in I$ and m_j for $j\in J$ such that*

$$\sigma_i=\ell_i T \quad \text{for } i\in I, \quad \text{and} \quad \tau_j=m_j T \quad \text{for } j\in J.$$

Define

$$\sigma=\max_{i\in I}\sigma_i, \quad \tau=\max_{j\in J}\tau_j \quad \text{and} \quad r=\max\{\sigma,\tau\}.$$

(σ, τ and r are positive integers.)

As usual, by a *solution* of the difference equation (E) we mean a sequence $(x_n)_{n\geqslant -r}$ of real numbers, which satisfies (E) for all integers $n\geqslant 0$.

For convenience, let us introduce the set Φ defined by

$$\Phi=\{\phi=(\phi_n)_{n=-r}^{0}:\ \phi_n\in\mathbf{R} \quad \text{for } n=-r,\dots,0\}.$$

This set is a Banach space with the usual sup-norm $\|\cdot\|$ defined as follows

$$\|\phi\|=\sup_{n=-r,\dots,0}|\phi_n| \quad \text{for each } \phi=(\phi_n)_{n=-r}^{0} \text{ in } \Phi.$$

It is clear that, for any $\phi=(\phi_n)_{n=-r}^{0}$ in Φ, there exists a unique solution $(x_n)_{n\geqslant -r}$ of the neutral delay difference equation (E) which satisfies the *initial condition*

$$x_n=\phi_n \quad \text{for } n=-r,\dots,0; \tag{C}$$

this solution is said to be the solution of the *initial problem* (E)–(C) or, more briefly, the solution of (E)–(C).

The equation

$$\left[\lambda\left(1+\sum_{i\in I} c_i\lambda^{-\sigma_i}\right)\right]^T = \prod_{k=0}^{T-1}\left[1+\sum_{i\in I} c_i\lambda^{-\sigma_i} + a(k) + \sum_{j\in J} b_j(k)\lambda^{-\tau_j}\right] \tag{*}$$

is associated with the neutral delay difference equation (E) and will be called the *characteristic equation* of (E).

The results of the paper are obtained by the use of a positive root of the characteristic equation (*), which has some suitable properties.

It will be considered that the reader is familiar with the notions of the *stability*, *instability*, and *asymptotic stability* of the *trivial solution* of the neutral delay difference equation (E).

The main results of this paper are two theorems (Theorems 1 and 2) and two corollaries (Corollaries 1 and 2) of the first of these theorems, and will be stated in Section 2. Theorem 1 establishes a basic asymptotic criterion for the solutions of the difference equation (E). The application of Theorem 1 to a specific case leads to Corollary 1, while Corollary 2 constitutes a nonoscillation result for (E). Theorem 2 gives an estimate of the solutions of (E) and provides sufficient conditions for the stability, the asymptotic stability and the instability of the trivial solution of (E).

The proofs of Theorems 1 and 2 will be given in Sections 3 and 4, respectively.

Section 5 contains a discussion on the application of the main results to the special case of (nonneutral) delay difference equations with periodic coefficients and constant delays as well as to the special case of neutral delay difference equations with constant coefficients and constant delays.

2. Statement of the main results

In order to state the main results of the paper, we first introduce certain notations. These notations will be used throughout the paper without any further mention.

By $(\widetilde{a}(n))_{n\geqslant -r}$ and $(\widetilde{b}_j(n))_{n\geqslant -r}$ for $j\in J$ we will denote the T-periodic extensions of the coefficients $(a(n))_{n\geqslant 0}$ and $(b_j(n))_{n\geqslant 0}$ for $j\in J$ respectively. (Note that r is obviously a multiple of the period T.)

We consider positive roots λ_0 of the characteristic equation (*) with the property

$$1+\sum_{i\in I} c_i\lambda_0^{-\sigma_i} > 0. \tag{$\mathrm{p}_0(\lambda_0)$}$$

Such a root λ_0 of (*) satisfies

$$\left[\lambda_0\left(1+\sum_{i\in I} c_i\lambda_0^{-\sigma_i}\right)\right]^T = \prod_{k=0}^{T-1}\left[1+\sum_{i\in I} c_i\lambda_0^{-\sigma_i} + a(k) + \sum_{j\in J} b_j(k)\lambda_0^{-\tau_j}\right],$$

which can equivalently be written

$$\lambda_0^T = \prod_{k=0}^{T-1}\left\{1+\frac{1}{1+\sum_{i\in I} c_i\lambda_0^{-\sigma_i}}\left[a(k)+\sum_{j\in J} b_j(k)\lambda_0^{-\tau_j}\right]\right\}.$$

Furthermore, if λ_0 is a positive root of the characteristic equation (*) with the property ($\mathrm{p}_0(\lambda_0)$), then $(h_{\lambda_0}(n))_{n\geqslant -r}$ will stand for the sequence of real numbers defined as follows

$$h_{\lambda_0}(n) = 1 + \frac{1}{1+\sum_{i\in I} c_i \lambda_0^{-\sigma_i}} \left[\widetilde{a}(n) + \sum_{j\in J} \widetilde{b}_j(n) \lambda_0^{-\tau_j} \right] \quad \text{for } n \geqslant -r.$$

It follows immediately that the T-periodicity of the coefficients $(a(n))_{n\geqslant 0}$ and $(b_j(n))_{n\geqslant 0}$ for $j \in J$ guarantees that, for any positive root λ_0 of (*) with the property ($\mathrm{p}_0(\lambda_0)$), the sequence $(h_{\lambda_0}(n))_{n\geqslant -r}$ is also T-periodic.

By using the last notation, we have

$$\lambda_0^T = \prod_{k=0}^{T-1} h_{\lambda_0}(k)$$

for each positive root λ_0 of the characteristic equation (*) with the property ($\mathrm{p}_0(\lambda_0)$). This fact will be quite frequently used in the sequel without any specific mention.

We need positive roots λ_0 of the characteristic equation (*) with the property ($\mathrm{p}_0(\lambda_0)$) and the additional property:

($\mathrm{p}(\lambda_0)$) *If* $T > 1$, *then*

$$h_{\lambda_0}(k) \equiv 1 + \frac{1}{1+\sum_{i\in I} c_i \lambda_0^{-\sigma_i}} \left[a(k) + \sum_{j\in J} b_j(k) \lambda_0^{-\tau_j} \right] > 0 \quad (k = 1, \ldots, T-1).$$

(Clearly, ($\mathrm{p}(\lambda_0)$) holds by itself when $T = 1$.)

We give here a simple result, which we will keep in mind in what follows:

If λ_0 *is a positive root of the characteristic equation* (*) *with the properties* ($\mathrm{p}_0(\lambda_0)$) *and* ($\mathrm{p}(\lambda_0)$), *then we have*

$$h_{\lambda_0}(n) > 0 \quad \textit{for all } n \geqslant -r.$$

To prove this result, we observe that for $T = 1$ it holds

$$h_{\lambda_0}(0) = \lambda_0 > 0.$$

If $T > 1$, then from ($\mathrm{p}(\lambda_0)$) it follows that

$$h_{\lambda_0}(k) > 0 \quad \text{for } k = 1, \ldots, T-1$$

and hence

$$h_{\lambda_0}(0) = \frac{\lambda_0^T}{\prod_{k=1}^{T-1} h_{\lambda_0}(k)} > 0.$$

We have thus seen that, in both cases where $T = 1$ or $T > 1$, it holds

$$h_{\lambda_0}(k) > 0 \quad \text{for } k = 0, 1, \ldots, T-1.$$

Since the sequences $(\widetilde{a}(n))_{n\geqslant -r}$ and $(\widetilde{b}_j(n))_{n\geqslant -r}$ for $j\in J$ are T-periodic and r is a multiple of the period T, it follows immediately that $h_{\lambda_0}(n)>0$ for all integers $n\geqslant -r$.

To obtain the results of the paper, we will make use of a positive root λ_0 of the characteristic equation (*) with the properties ($\mathrm{p}_0(\lambda_0)$) and ($\mathrm{p}(\lambda_0)$) as well as with the following property:

$$\sum_{i\in I}|c_i|\left[1+\sum_{s=0}^{\sigma_i-1}\left|1-\frac{1}{h_{\lambda_0}(s)}\right|\right]\lambda_0^{-\sigma_i}+\sum_{j\in J}\left[\sum_{s=0}^{\tau_j-1}\frac{1}{h_{\lambda_0}(s)}|b_j(s)|\right]\lambda_0^{-\tau_j}<1. \tag{P(λ_0)}$$

For our convenience, for any positive root λ_0 of the characteristic equation (*) with the properties ($\mathrm{p}_0(\lambda_0)$) and ($\mathrm{p}(\lambda_0)$), let us define

$$H_{\lambda_0}(n)=\begin{cases}\prod_{k=0}^{n-1}h_{\lambda_0}(k), & \text{for } n\geqslant 0\\ \left[\prod_{k=n}^{-1}h_{\lambda_0}(k)\right]^{-1}, & \text{for } n=-r,\ldots,0.\end{cases}$$

Note that, in this paper, we use the usual convention

$$\prod_{k=0}^{-1}=1.$$

Now, we are in a position to state the main results of the paper.

Theorem 1. *Let λ_0 be a positive root of the characteristic equation* (*) *with the properties* ($\mathrm{p}_0(\lambda_0)$), ($\mathrm{p}(\lambda_0)$) *and* ($\mathrm{P}(\lambda_0)$). *Set*

$$\gamma_{\lambda_0}=\sum_{i\in I}c_i\left\{1-\sum_{s=0}^{\sigma_i-1}\left[1-\frac{1}{h_{\lambda_0}(s)}\right]\right\}\lambda_0^{-\sigma_i}+\sum_{j\in J}\left[\sum_{s=0}^{\tau_j-1}\frac{1}{h_{\lambda_0}(s)}b_j(s)\right]\lambda_0^{-\tau_j}.$$

Then, for any $\phi=(\phi_n)_{n=-r}^{0}$ in Φ, the solution $(x_n)_{n\geqslant -r}$ of (E)–(C) *satisfies*

$$\lim_{n\to\infty}\frac{x_n}{H_{\lambda_0}(n)}=\frac{L_{\lambda_0}(\phi)}{1+\gamma_{\lambda_0}},$$

where

$$\begin{aligned}L_{\lambda_0}(\phi)=\phi_0&+\sum_{i\in I}c_i\left(\phi_{-\sigma_i}-\left\{\sum_{s=-\sigma_i}^{-1}\left[1-\frac{1}{h_{\lambda_0}(s)}\right]\left[\prod_{k=s}^{-1}h_{\lambda_0}(k)\right]\phi_s\right\}\lambda_0^{-\sigma_i}\right)\\&+\sum_{j\in J}\left\{\sum_{s=-\tau_j}^{-1}\left[\prod_{k=s+1}^{-1}h_{\lambda_0}(k)\right]\widetilde{b}_j(s)\phi_s\right\}\lambda_0^{-\tau_j}.\end{aligned}$$

Note: Property ($\mathrm{P}(\lambda_0)$) guarantees that $1+\gamma_{\lambda_0}>0$.

Let us assume that

$$a(n)+\sum_{j\in J} b_j(n)=0 \quad \textit{for } n\geqslant 0 \tag{Q$_1$}$$

and

$$\sum_{i\in I}|c_i|+\sum_{j\in J}\left[\sum_{s=0}^{\tau_j-1}|b_j(s)|\right]<1. \tag{Q$_2$}$$

From (Q_1) it follows immediately that $\lambda_0=1$ is a (positive) root of the characteristic equation (*). Assumption (Q_2) guarantees that $1+\sum_{i\in I} c_i>0$ and so the root $\lambda_0=1$ of (*) has the property ($p_0(\lambda_0)$). Moreover, by using again the assumption (Q_1), we immediately see that, for the root $\lambda_0=1$ of (*), we have $h_{\lambda_0}(n)=1$ for $n\geqslant -r$. Furthermore, we observe that the root $\lambda_0=1$ of (*) admits the property ($p(\lambda_0)$). Also, this root has the property ($P(\lambda_0)$), since we have assumed that (Q_2) holds. Thus, an application of Theorem 1 with $\lambda_0=1$ leads to the following corollary.

Corollary 1. *Assume that* (Q_1) *and* (Q_2) *hold. Then, for any* $\phi=(\phi_n)_{n=-r}^0$ *in* Φ, *the solution* $(x_n)_{n\geqslant -r}$ *of* (E)–(C) *satisfies*

$$\lim_{n\to\infty} x_n=\frac{\phi_0+\sum_{i\in I} c_i\phi_{-\sigma_i}+\sum_{j\in J}\left[\sum_{s=-\tau_j}^{-1}\widetilde{b}_j(s)\phi_s\right]}{1+\sum_{i\in I} c_i+\sum_{j\in J}\left[\sum_{s=0}^{\tau_j-1} b_j(s)\right]}.$$

Note: Assumption (Q_2) guarantees that

$$1+\sum_{i\in I} c_i+\sum_{j\in J}\left[\sum_{s=0}^{\tau_j-1} b_j(s)\right]>0.$$

Another interesting consequence of Theorem 1 is the following corollary, which constitutes a *nonoscillation* criterion for the solutions of the neutral delay difference equation (E).

Corollary 2. *Let* λ_0 *be a positive root of the characteristic equation* (*) *with the properties* ($p_0(\lambda_0)$), ($p(\lambda_0)$) *and* ($P(\lambda_0)$). *Then, for any* $\phi=(\phi_n)_{n=-r}^0$ *in* Φ, *the solution* $(x_n)_{n\geqslant -r}$ *of* (E)–(C) *will be nonoscillatory, except possibly if* ϕ *is such that* $L_{\lambda_0}(\phi)=0$, *where* $L_{\lambda_0}(\phi)$ *is defined as in Theorem* 1.

Consider a positive root λ_0 of (*) with the properties ($p_0(\lambda_0)$), ($p(\lambda_0)$) and ($P(\lambda_0)$). Moreover, for any $\phi=(\phi_n)_{n=-r}^0$ in Φ, let $L_{\lambda_0}(\phi)$ be defined as in Theorem 1. Clearly, the operator L_{λ_0} is linear. Furthermore, there exists $\phi^0=(\phi_n^0)_{n=-r}^0$ in Φ such that $L_{\lambda_0}(\phi^0)\neq 0$. Indeed, if we set

$$\phi_n^0=\left[\prod_{k=n}^{-1} h_{\lambda_0}(k)\right]^{-1} \quad \text{for } n=-r,\ldots,0,$$

then $\phi^0=(\phi_n^0)_{n=-r}^0$ belongs to Φ and we have

$$\begin{aligned}
L_{\lambda_0}(\phi^0)=&1+\sum_{i\in I}c_i\left(\left[\prod_{k=-\sigma_i}^{-1}h_{\lambda_0}(k)\right]^{-1}-\left\{\sum_{s=-\sigma_i}^{-1}\left[1-\frac{1}{h_{\lambda_0}(s)}\right]\left[\prod_{k=s}^{-1}h_{\lambda_0}(k)\right]\cdot\right.\right.\\
&\left.\left.\cdot\left[\prod_{k=s}^{-1}h_{\lambda_0}(k)\right]^{-1}\right\}\lambda_0^{-\sigma_i}\right)\\
&+\sum_{j\in J}\left\{\sum_{s=-\tau_j}^{-1}\left[\prod_{k=s+1}^{-1}h_{\lambda_0}(k)\right]\widetilde{b}_j(s)\left[\prod_{k=s}^{-1}h_{\lambda_0}(k)\right]^{-1}\right\}\lambda_0^{-\tau_j}\\
=&1+\sum_{i\in I}c_i\left(\left[\prod_{k=-\sigma_i}^{-1}h_{\lambda_0}(k)\right]^{-1}-\left\{\sum_{s=-\sigma_i}^{-1}\left[1-\frac{1}{h_{\lambda_0}(s)}\right]\right\}\lambda_0^{-\sigma_i}\right)\\
&+\sum_{j\in J}\left[\sum_{s=-\tau_j}^{-1}\frac{1}{h_{\lambda_0}(s)}\widetilde{b}_j(s)\right]\lambda_0^{-\tau_j}.
\end{aligned}$$

But, if i is an arbitrary index in I, then we can take into account the fact that the sequence $(h_{\lambda_0}(n))_{n\geqslant -r}$ is T-periodic and that $\sigma_i=\ell_i T$ to obtain

$$\prod_{k=-\sigma_i}^{-1}h_{\lambda_0}(k)=\prod_{k=0}^{\sigma_i-1}h_{\lambda_0}(k)=\left[\prod_{k=0}^{T-1}h_{\lambda_0}(k)\right]^{\ell_i}=(\lambda_0^T)^{\ell_i}=\lambda_0^{\ell_i T}=\lambda_0^{\sigma_i}$$

and

$$\sum_{s=-\sigma_i}^{-1}\left[1-\frac{1}{h_{\lambda_0}(s)}\right]=\sum_{s=0}^{\sigma_i-1}\left[1-\frac{1}{h_{\lambda_0}(s)}\right].$$

Moreover, for each index $j\in J$, the T-periodicity of the sequences $(h_{\lambda_0}(n))_{n\geqslant -r}$ and $(\widetilde{b}_j(n))_{n\geqslant -r}$ and the fact that $\tau_j=m_jT$ imply immediately that

$$\sum_{s=-\tau_j}^{-1}\frac{1}{h_{\lambda_0}(s)}\widetilde{b}_j(s)=\sum_{s=0}^{\tau_j-1}\frac{1}{h_{\lambda_0}(s)}b_j(s).$$

So, we can find

$$\begin{aligned}
L_{\lambda_0}(\phi^0)=&1+\sum_{i\in I}c_i\left\{1-\sum_{s=0}^{\sigma_i-1}\left[1-\frac{1}{h_{\lambda_0}(s)}\right]\right\}\lambda_0^{-\sigma_i}+\sum_{j\in J}\left[\sum_{s=0}^{\tau_j-1}\frac{1}{h_{\lambda_0}(s)}b_j(s)\right]\lambda_0^{-\tau_j}\\
\equiv&1+\gamma_{\lambda_0}>0,
\end{aligned}$$

where γ_{λ_0} is defined as in Theorem 1. Hence, by the same method with the one that was used by Kordonis et al. [14], one can prove the following result (which can be considered as a complement of Corollary 2):

Let λ_0 *be a positive root of the characteristic equation* (*) *with the properties* ($\mathrm{p}_0(\lambda_0)$), ($\mathrm{p}(\lambda_0)$) *and* ($\mathrm{P}(\lambda_0)$). *Moreover, for any* $\phi = (\phi_n)_{n=-r}^{0}$ *in* Φ, *let* $L_{\lambda_0}(\phi)$ *be defined as in Theorem* 1. *Then the set of all* $\phi = (\phi_n)_{n=-r}^{0}$ *in* Φ *which satisfy* $L_{\lambda_0}(\phi) = 0$ *is a nowhere dense subset of the Banach space* Φ (*with the sup-norm*).

Theorem 2. *Let* λ_0 *be a positive root of the characteristic equation* (*) *with the properties* ($\mathrm{p}_0(\lambda_0)$), ($\mathrm{p}(\lambda_0)$) *and* ($\mathrm{P}(\lambda_0)$).

Consider γ_{λ_0} *as in Theorem* 1 *and set*

$$\mu_{\lambda_0} = \sum_{i \in I} |c_i| \left[1 + \sum_{s=0}^{\sigma_i - 1} \left| 1 - \frac{1}{h_{\lambda_0}(s)} \right| \right] \lambda_0^{-\sigma_i} + \sum_{j \in J} \left[\sum_{s=0}^{\tau_j - 1} \frac{1}{h_{\lambda_0}(s)} |b_j(s)| \right] \lambda_0^{-\tau_j}.$$

Then, for any $\phi = (\phi_n)_{n=-r}^{0}$ *in* Φ, *the solution* $(x_n)_{n \geqslant -r}$ *of* (E)–(C) *satisfies*

$$|x_n| \leqslant N_{\lambda_0} \parallel \phi \parallel H_{\lambda_0}(n) \quad \textit{for all } n \geqslant 0,$$

where

$$N_{\lambda_0} = \frac{1 + \mu_{\lambda_0}}{1 + \gamma_{\lambda_0}} + \mu_{\lambda_0} \left(1 + \frac{1 + \mu_{\lambda_0}}{1 + \gamma_{\lambda_0}} \right) \max_{s=-r,\dots,0} \prod_{k=s}^{-1} h_{\lambda_0}(k).$$

The constant N_{λ_0} *is greater than* 1.

Moreover, the trivial solution of the difference equation (E) *is*:

(i) *Stable* (*at* 0) *if*

$$\limsup_{n \to \infty} H_{\lambda_0}(n) < \infty. \tag{$\mathrm{G}_1(\lambda_0)$}$$

(ii) *Asymptotically stable* (*at* 0) *if*

$$\lim_{n \to \infty} H_{\lambda_0}(n) = 0. \tag{$\mathrm{G}_2(\lambda_0)$}$$

(iii) *Unstable* (*at* 0) *if*

$$\limsup_{n \to \infty} H_{\lambda_0}(n) = \infty. \tag{$\mathrm{G}_3(\lambda_0)$}$$

Let (Q_1) and (Q_2) be satisfied. Then, as we have previously seen, $\lambda_0 = 1$ is a (positive) root of the characteristic equation (*) with the properties ($\mathrm{p}_0(\lambda_0)$), ($\mathrm{p}(\lambda_0)$) and ($\mathrm{P}(\lambda_0)$) and such that $h_{\lambda_0}(n) = 1$ for $n \geqslant -r$ (and so $H_{\lambda_0}(n) = 1$ for all $n \geqslant -r$). Hence, from the stability criterion (i) in Theorem 2 we can, in particular, obtain the following result:

The trivial solution of (E) *is stable* (*at* 0) *if* (Q_1) *and* (Q_2) *hold.*

3. Proof of Theorem 1

First of all, let us define μ_{λ_0} as in Theorem 2. We immediately see that the property (P(λ_0)) guarantees that

$$0<\mu_{\lambda_0}<1.$$

Furthermore, we have $|\gamma_{\lambda_0}|\leqslant\mu_{\lambda_0}$. So, it holds $|\gamma_{\lambda_0}|<1$, which implies that

$$1+\gamma_{\lambda_0}>0.$$

Next, we will give some equalities needed below. These equalities are consequences of the fact that the sequences $(h_{\lambda_0}(n))_{n\geqslant -r}$ and $(\widetilde{b}_j(n))_{n\geqslant -r}$ for $j\in J$ are T-periodic and that $\sigma_i=\ell_i T$ for $i\in I$ and $\tau_j=m_jT$ for $j\in J$. First, it is a matter of elementary calculations to prove that

$$H_{\lambda_0}(n-\sigma_i)=\lambda_0^{-\sigma_i}H_{\lambda_0}(n)\quad \text{for every } n\geqslant 0\quad \text{and all } i\in I. \tag{3.1}$$

In a similar manner, one can verify that

$$H_{\lambda_0}(n-\tau_j)=\lambda_0^{-\tau_j}H_{\lambda_0}(n)\quad \text{for every } n\geqslant 0\quad \text{and all } j\in J. \tag{3.2}$$

Furthermore, it follows immediately that

$$\sum_{s=n-\sigma_i}^{n-1}\left[1-\frac{1}{h_{\lambda_0}(s)}\right]=\sum_{s=0}^{\sigma_i-1}\left[1-\frac{1}{h_{\lambda_0}(s)}\right]\quad \text{for all } n\geqslant 0 \text{ and } i\in I, \tag{3.3}$$

$$\sum_{s=n-\sigma_i}^{n-1}\left|1-\frac{1}{h_{\lambda_0}(s)}\right|=\sum_{s=0}^{\sigma_i-1}\left|1-\frac{1}{h_{\lambda_0}(s)}\right|\quad \text{for all } n\geqslant 0 \text{ and } i\in I. \tag{3.4}$$

Moreover, we have

$$\sum_{s=n-\tau_j}^{n-1}\frac{1}{h_{\lambda_0}(s)}\widetilde{b}_j(s)=\sum_{s=0}^{\tau_j-1}\frac{1}{h_{\lambda_0}(s)}b_j(s)\quad \text{for all } n\geqslant 0 \text{ and } j\in J, \tag{3.5}$$

$$\sum_{s=n-\tau_j}^{n-1}\frac{1}{h_{\lambda_0}(s)}|\widetilde{b}_j(s)|=\sum_{s=0}^{\tau_j-1}\frac{1}{h_{\lambda_0}(s)}|b_j(s)|\ \text{for all } n\geqslant 0 \text{ and } j\in J. \tag{3.6}$$

Now, let us consider an arbitrary $\phi=(\phi_n)_{n=-r}^{0}$ in Φ and let $(x_n)_{n\geqslant -r}$ be the solution of (E)–(C). Define

$$y_n=\frac{x_n}{H_{\lambda_0}(n)}\quad \text{for } n\geqslant -r.$$

Then, by taking into account (3.1) and (3.2), we obtain for every $n \geqslant 0$

$$
\begin{aligned}
&\Delta\left(x_n + \sum_{i\in I} c_i x_{n-\sigma_i}\right) - a(n)x_n - \sum_{j\in J} b_j(n)x_{n-\tau_j} \\
&\quad = \Delta\left[H_{\lambda_0}(n)y_n + \sum_{i\in I} c_i H_{\lambda_0}(n-\sigma_i)y_{n-\sigma_i}\right] - a(n)H_{\lambda_0}(n)y_n \\
&\qquad - \sum_{j\in J} b_j(n)H_{\lambda_0}(n-\tau_j)y_{n-\tau_j} \\
&\quad = \Delta\left[H_{\lambda_0}(n)\left(y_n + \sum_{i\in I} c_i\lambda_0^{-\sigma_i}y_{n-\sigma_i}\right)\right] - a(n)H_{\lambda_0}(n)y_n \\
&\qquad - H_{\lambda_0}(n)\sum_{j\in J} b_j(n)\lambda_0^{-\tau_j}y_{n-\tau_j} \\
&\quad = H_{\lambda_0}(n)\left\{h_{\lambda_0}(n)\Delta\left(y_n + \sum_{i\in I} c_i\lambda_0^{-\sigma_i}y_{n-\sigma_i}\right) + \left[h_{\lambda_0}(n) - 1 - a(n)\right]y_n\right. \\
&\qquad \left. + \left[h_{\lambda_0}(n) - 1\right]\sum_{i\in I} c_i\lambda_0^{-\sigma_i}y_{n-\sigma_i} - \sum_{j\in J} b_j(n)\lambda_0^{-\tau_j}y_{n-\tau_j}\right\} \\
&\quad = H_{\lambda_0}(n)\left(h_{\lambda_0}(n)\Delta\left(y_n + \sum_{i\in I} c_i\lambda_0^{-\sigma_i}y_{n-\sigma_i}\right)\right. \\
&\qquad + \left\{-[h_{\lambda_0}(n) - 1]\sum_{i\in I} c_i\lambda_0^{-\sigma_i} + \sum_{j\in J} b_j(n)\lambda_0^{-\tau_j}\right\}y_n \\
&\qquad \left. + [h_{\lambda_0}(n) - 1]\sum_{i\in I} c_i\lambda_0^{-\sigma_i}y_{n-\sigma_i} - \sum_{j\in J} b_j(n)\lambda_0^{-\tau_j}y_{n-\tau_j}\right) \\
&\quad = H_{\lambda_0}(n)\left\{h_{\lambda_0}(n)\Delta\left(y_n + \sum_{i\in I} c_i\lambda_0^{-\sigma_i}y_{n-\sigma_i}\right)\right. \\
&\qquad - [h_{\lambda_0}(n) - 1]\sum_{i\in I} c_i\lambda_0^{-\sigma_i}(y_n - y_{n-\sigma_i}) \\
&\qquad \left. + \sum_{j\in J} b_j(n)\lambda_0^{-\tau_j}(y_n - y_{n-\tau_j})\right\}.
\end{aligned}
$$

Thus, the fact that $(x_n)_{n\geqslant -r}$ satisfies (E) for $n\geqslant 0$ is equivalent to the fact that $(y_n)_{n\geqslant -r}$ satisfies

$$\Delta\left(y_n+\sum_{i\in I}c_i\lambda_0^{-\sigma_i}y_{n-\sigma_i}\right)=\left[1-\frac{1}{h_{\lambda_0}(n)}\right]\sum_{i\in I}c_i\lambda_0^{-\sigma_i}(y_n-y_{n-\sigma_i})$$
$$-\frac{1}{h_{\lambda_0}(n)}\sum_{j\in J}b_j(n)\lambda_0^{-\tau_j}(y_n-y_{n-\tau_j})\quad\text{for } n\geqslant 0. \tag{3.7}$$

On the other hand, the initial condition (C) becomes

$$y_n=\frac{\phi_n}{H_{\lambda_0}(n)}\quad\text{for } n=-r,\ldots,0. \tag{3.8}$$

By taking into account the fact that the sequences $(h_{\lambda_0}(n))_{n\geqslant -r}$ and $(\widetilde{b}_j(n))_{n\geqslant -r}$ for $j\in J$ are T-periodic and that $\sigma_i=\ell_iT$ for $i\in I$ and $\tau_j=m_jT$ for $j\in J$, we derive for $n\geqslant 0$

$$\left[1-\frac{1}{h_{\lambda_0}(n)}\right]\sum_{i\in I}c_i\lambda_0^{-\sigma_i}(y_n-y_{n-\sigma_i})-\frac{1}{h_{\lambda_0}(n)}\sum_{j\in J}b_j(n)\lambda_0^{-\tau_j}(y_n-y_{n-\tau_j})$$
$$=\sum_{i\in I}c_i\left\{\left[1-\frac{1}{h_{\lambda_0}(n)}\right]y_n-\left[1-\frac{1}{h_{\lambda_0}(n-\sigma_i)}\right]y_{n-\sigma_i}\right\}\lambda_0^{-\sigma_i}$$
$$-\sum_{j\in J}\left[\frac{1}{h_{\lambda_0}(n)}\widetilde{b}_j(n)y_n-\frac{1}{h_{\lambda_0}(n-\tau_j)}\widetilde{b}_j(n-\tau_j)y_{n-\tau_j}\right]\lambda_0^{-\tau_j}$$
$$=\sum_{i\in I}c_i\left\{\Delta\sum_{s=n-\sigma_i}^{n-1}\left[1-\frac{1}{h_{\lambda_0}(s)}\right]y_s\right\}\lambda_0^{-\sigma_i}-\sum_{j\in J}\left[\Delta\sum_{s=n-\tau_j}^{n-1}\frac{1}{h_{\lambda_0}(s)}\widetilde{b}_j(s)y_s\right]\lambda_0^{-\tau_j}.$$

Therefore, (3.7) can equivalently be written

$$\Delta\left(y_n+\sum_{i\in I}c_i\lambda_0^{-\sigma_i}y_{n-\sigma_i}\right)=\Delta\left(\sum_{i\in I}c_i\left\{\sum_{s=n-\sigma_i}^{n-1}\left[1-\frac{1}{h_{\lambda_0}(s)}\right]y_s\right\}\lambda_0^{-\sigma_i}\right.$$
$$\left.-\sum_{j\in J}\left[\sum_{s=n-\tau_j}^{n-1}\frac{1}{h_{\lambda_0}(s)}\widetilde{b}_j(s)y_s\right]\lambda_0^{-\tau_j}\right)\quad\text{for } n\geqslant 0.$$

This is equivalent to

$$y_n+\sum_{i\in I}c_i\lambda_0^{-\sigma_i}y_{n-\sigma_i}=\sum_{i\in I}c_i\left\{\sum_{s=n-\sigma_i}^{n-1}\left[1-\frac{1}{h_{\lambda_0}(s)}\right]y_s\right\}\lambda_0^{-\sigma_i}$$
$$-\sum_{j\in J}\left[\sum_{s=n-\tau_j}^{n-1}\frac{1}{h_{\lambda_0}(s)}\widetilde{b}_j(s)y_s\right]\lambda_0^{-\tau_j}+K\quad\text{for } n\geqslant 0, \tag{3.9}$$

where

$$K=\left(y_0+\sum_{i\in I} c_i\lambda_0^{-\sigma_i}y_{-\sigma_i}\right)-\sum_{i\in I} c_i\left\{\sum_{s=-\sigma_i}^{-1}\left[1-\frac{1}{h_{\lambda_0}(s)}\right]y_s\right\}\lambda_0^{-\sigma_i}$$
$$+\sum_{j\in J}\left[\sum_{s=-\tau_j}^{-1}\frac{1}{h_{\lambda_0}(s)}\widetilde{b}_j(s)y_s\right]\lambda_0^{-\tau_j}.$$

But, by taking into account (3.1) (for $n=0$) and using (3.8), we get

$$K=\phi_0+\sum_{i\in I} c_i\lambda_0^{-\sigma_i}\frac{\phi_{-\sigma_i}}{H_{\lambda_0}(-\sigma_i)}-\sum_{i\in I} c_i\left\{\sum_{s=-\sigma_i}^{-1}\left[1-\frac{1}{h_{\lambda_0}(s)}\right]\frac{\phi_s}{H_{\lambda_0}(s)}\right\}\lambda_0^{-\sigma_i}$$
$$+\sum_{j\in J}\left[\sum_{s=-\tau_j}^{-1}\frac{1}{h_{\lambda_0}(s)}\widetilde{b}_j(s)\frac{\phi_s}{H_{\lambda_0}(s)}\right]\lambda_0^{-\tau_j}$$
$$=\phi_0+\sum_{i\in I} c_i\left(\phi_{-\sigma_i}-\left\{\sum_{s=-\sigma_i}^{-1}\left[1-\frac{1}{h_{\lambda_0}(s)}\right]\frac{1}{H_{\lambda_0}(s)}\phi_s\right\}\lambda_0^{-\sigma_i}\right)$$
$$+\sum_{j\in J}\left[\sum_{s=-\tau_j}^{-1}\frac{1}{h_{\lambda_0}(s)H_{\lambda_0}(s)}\widetilde{b}_j(s)\phi_s\right]\lambda_0^{-\tau_j}$$
$$=\phi_0+\sum_{i\in I} c_i\left(\phi_{-\sigma_i}-\left\{\sum_{s=-\sigma_i}^{-1}\left[1-\frac{1}{h_{\lambda_0}(s)}\right]\left[\prod_{k=s}^{-1}h_{\lambda_0}(k)\right]\phi_s\right\}\lambda_0^{-\sigma_i}\right)$$
$$+\sum_{j\in J}\left\{\sum_{s=-\tau_j}^{-1}\left[\prod_{k=s+1}^{-1}h_{\lambda_0}(k)\right]\widetilde{b}_j(s)\phi_s\right\}\lambda_0^{-\tau_j}.$$

So, by the definition of $L_{\lambda_0}(\phi)$, we have $K\equiv L_{\lambda_0}(\phi)$. Hence, (3.9) becomes

$$y_n+\sum_{i\in I} c_i\lambda_0^{-\sigma_i}y_{n-\sigma_i}=\sum_{i\in I} c_i\left\{\sum_{s=n-\sigma_i}^{n-1}\left[1-\frac{1}{h_{\lambda_0}(s)}\right]y_s\right\}\lambda_0^{-\sigma_i}$$
$$-\sum_{j\in J}\left[\sum_{s=n-\tau_j}^{n-1}\frac{1}{h_{\lambda_0}(s)}\widetilde{b}_j(s)y_s\right]\lambda_0^{-\tau_j}+L_{\lambda_0}(\phi)\quad\text{for } n\geqslant 0. \tag{3.10}$$

Next, we set

$$z_n=y_n-\frac{L_{\lambda_0}(\phi)}{1+\gamma_{\lambda_0}}\quad\text{for } n\geqslant -r.$$

Then, by using (3.3) and (3.5) and taking into account the definition of γ_{λ_0}, we can easily verify that (3.10) takes the following equivalent form

$$\begin{aligned} z_n + \sum_{i\in I} c_i \lambda_0^{-\sigma_i} z_{n-\sigma_i} = & \sum_{i\in I} c_i \left\{ \sum_{s=n-\sigma_i}^{n-1} \left[1 - \frac{1}{h_{\lambda_0}(s)}\right] z_s \right\} \lambda_0^{-\sigma_i} \\ & - \sum_{j\in J} \left[\sum_{s=n-\tau_j}^{n-1} \frac{1}{h_{\lambda_0}(s)} \widetilde{b}_j(s) z_s \right] \lambda_0^{-\tau_j} \quad \text{for } n \geqslant 0. \end{aligned} \tag{3.11}$$

Moreover, (3.8) reduces to

$$z_n = \frac{\phi_n}{H_{\lambda_0}(n)} - \frac{L_{\lambda_0}(\phi)}{1+\gamma_{\lambda_0}} \quad \text{for } n = -r, \dots, 0. \tag{3.12}$$

Because of the definitions of $(y_n)_{n\geqslant -r}$ and $(z_n)_{n\geqslant -r}$, the proof will be accomplished by proving that

$$\lim_{n\to\infty} z_n = 0. \tag{3.13}$$

In the rest of the proof we will establish (3.13).

Set

$$M_{\lambda_0}(\phi) = \max_{n=-r,\dots,0} \left| \frac{\phi_n}{H_{\lambda_0}(n)} - \frac{L_{\lambda_0}(\phi)}{1+\gamma_{\lambda_0}} \right|. \tag{3.14}$$

In view of (3.12), from (3.14) it follows that

$$|z_n| \leqslant M_{\lambda_0}(\phi) \quad \text{for } n = -r, \dots, 0. \tag{3.15}$$

We will show that

$$|z_n| \leqslant M_{\lambda_0}(\phi) \quad \text{for all } n \geqslant -r. \tag{3.16}$$

For this purpose, let us consider an arbitrary number $\varepsilon > 0$. We claim that

$$|z_n| < M_{\lambda_0}(\phi) + \varepsilon \quad \text{for every } n \geqslant -r. \tag{3.17}$$

Otherwise, because of (3.15), there exists an integer $n_0 > 0$ so that

$$|z_n| < M_{\lambda_0}(\phi) + \varepsilon \quad \text{for } n = -r, \dots, n_0 - 1, \quad \text{and} \quad |z_{n_0}| \geqslant M_{\lambda_0}(\phi) + \varepsilon.$$

Thus, by using (3.4) and (3.6) and taking into account the fact that $\mu_{\lambda_0} < 1$, from (3.11) we obtain

$$\begin{aligned}
M_{\lambda_0}(\phi) + \varepsilon \leqslant |z_{n_0}| &\leqslant \sum_{i\in I} |c_i| \left[|z_{n_0-\sigma_i}| + \sum_{s=n_0-\sigma_i}^{n_0-1} \left| 1 - \frac{1}{h_{\lambda_0}(s)} \right| |z_s| \right] \lambda_0^{-\sigma_i} \\
&\quad + \sum_{j\in J} \left[\sum_{s=n_0-\tau_j}^{n_0-1} \frac{1}{h_{\lambda_0}(s)} |\widetilde{b}_j(s)||z_s| \right] \lambda_0^{-\tau_j} \\
&\leqslant \left\{ \sum_{i\in I} |c_i| \left[1 + \sum_{s=n_0-\sigma_i}^{n_0-1} \left| 1 - \frac{1}{h_{\lambda_0}(s)} \right| \right] \lambda_0^{-\sigma_i} \right. \\
&\quad \left. + \sum_{j\in J} \left[\sum_{s=n_0-\tau_j}^{n_0-1} \frac{1}{h_{\lambda_0}(s)} |\widetilde{b}_j(s)| \right] \lambda_0^{-\tau_j} \right\} [M_{\lambda_0}(\phi) + \varepsilon] \\
&= \left\{ \sum_{i\in I} |c_i| \left[1 + \sum_{s=0}^{\sigma_i-1} \left| 1 - \frac{1}{h_{\lambda_0}(s)} \right| \right] \lambda_0^{-\sigma_i} \right. \\
&\quad \left. + \sum_{j\in J} \left[\sum_{s=0}^{\tau_j-1} \frac{1}{h_{\lambda_0}(s)} \left| b_j(s) \right| \right] \lambda_0^{-\tau_j} \right\} [M_{\lambda_0}(\phi) + \varepsilon] \\
&= \mu_{\lambda_0} [M_{\lambda_0}(\phi) + \varepsilon] < M_{\lambda_0}(\phi) + \varepsilon.
\end{aligned}$$

We have arrived at a contradiction, which establishes our claim, i.e. (3.17) holds true. As (3.17) is fulfilled for all $\varepsilon > 0$, (3.16) is always satisfied. Furthermore, by (3.4), (3.6) and (3.16) as well as by the definition of μ_{λ_0}, from (3.11) we derive for each $n \geqslant 0$

$$\begin{aligned}
|z_n| &\leqslant \sum_{i\in I} |c_i| \left[|z_{n-\sigma_i}| + \sum_{s=n-\sigma_i}^{n-1} \left| 1 - \frac{1}{h_{\lambda_0}(s)} \right| |z_s| \right] \lambda_0^{-\sigma_i} \\
&\quad + \sum_{j\in J} \left[\sum_{s=n-\tau_j}^{n-1} \frac{1}{h_{\lambda_0}(s)} |\widetilde{b}_j(s)||z_s| \right] \lambda_0^{-\tau_j} \\
&\leqslant \left\{ \sum_{i\in I} |c_i| \left[1 + \sum_{s=n-\sigma_i}^{n-1} \left| 1 - \frac{1}{h_{\lambda_0}(s)} \right| \right] \lambda_0^{-\sigma_i} \right. \\
&\quad \left. + \sum_{j\in J} \left[\sum_{s=n-\tau_j}^{n-1} \frac{1}{h_{\lambda_0}(s)} |\widetilde{b}_j(s)| \right] \lambda_0^{-\tau_j} \right\} M_{\lambda_0}(\phi) \\
&= \left\{ \sum_{i\in I} |c_i| \left[1 + \sum_{s=0}^{\sigma_i-1} \left| 1 - \frac{1}{h_{\lambda_0}(s)} \right| \right] \lambda_0^{-\sigma_i} \right. \\
&\quad \left. + \sum_{j\in J} \left[\sum_{s=0}^{\tau_j-1} \frac{1}{h_{\lambda_0}(s)} \left| b_j(s) \right| \right] \lambda_0^{-\tau_j} \right\} M_{\lambda_0}(\phi) \\
&= \mu_{\lambda_0} M_{\lambda_0}(\phi),
\end{aligned}$$

i.e.,

$$|z_n| \leqslant \mu_{\lambda_0} M_{\lambda_0}(\phi) \quad \text{for all } n \geqslant 0. \tag{3.18}$$

By using (3.4), (3.6) and (3.11) and taking into account (3.16) and (3.18) as well as the definition of μ_{λ_0}, it is not difficult to establish, by an easy induction, that the sequence $(z_n)_{n \geqslant -r}$ satisfies

$$|z_n| \leqslant (\mu_{\lambda_0})^\rho M_{\lambda_0}(\phi) \quad \text{for } n \geqslant \rho r - r \quad (\rho = 0, 1, 2, \ldots). \tag{3.19}$$

Since $0 < \mu_{\lambda_0} < 1$, we have $\lim_{\rho \to \infty} (\mu_{\lambda_0})^\rho = 0$. Thus, from (3.19) it follows easily that $\lim_{n \to \infty} z_n = 0$, i.e., (3.13) holds true.

The proof of the theorem is now complete.

4. Proof of Theorem 2

By (P(λ_0)), it holds $0 < \mu_{\lambda_0} < 1$. We also have $|\gamma_{\lambda_0}| \leqslant \mu_{\lambda_0}$. So, it follows easily that $1 + \gamma_{\lambda_0} > 0$ and $N_{\lambda_0} > 1$. Consider now an arbitrary $\phi = (\phi_n)_{n=-r}^0$ in Φ and let $(x_n)_{n \geqslant -r}$ be the solution of (E)–(C). Let also $L_{\lambda_0}(\phi)$ be defined as in Theorem 1. Moreover, let $(y_n)_{n \geqslant -r}$ and $(z_n)_{n \geqslant -r}$ be defined as in the proof of Theorem 1. Then, as in the proof of Theorem 1, we arrive at (3.18), namely

$$|z_n| \leqslant \mu_{\lambda_0} M_{\lambda_0}(\phi) \quad \text{for all } n \geqslant 0, \tag{4.1}$$

where $M_{\lambda_0}(\phi)$ is defined by (3.14) (in the proof of Theorem 1), i.e.,

$$M_{\lambda_0}(\phi) = \max_{n=-r,\ldots,0} \left| \frac{\phi_n}{H_{\lambda_0}(n)} - \frac{L_{\lambda_0}(\phi)}{1 + \gamma_{\lambda_0}} \right|. \tag{4.2}$$

By the definition of $(z_n)_{n \geqslant -r}$, from (4.1) we obtain

$$|y_n| \leqslant \mu_{\lambda_0} M_{\lambda_0}(\phi) + \frac{|L_{\lambda_0}(\phi)|}{1 + \gamma_{\lambda_0}} \quad \text{for every } n \geqslant 0.$$

On the other hand, (4.2) gives

$$M_{\lambda_0}(\phi) \leqslant \| \phi \| \max_{n=-r,\ldots,0} [1/H_{\lambda_0}(n)] + \frac{|L_{\lambda_0}(\phi)|}{1 + \gamma_{\lambda_0}} = \| \phi \| \max_{s=-r,\ldots,0} \prod_{k=s}^{-1} h_{\lambda_0}(k) + \frac{|L_{\lambda_0}(\phi)|}{1 + \gamma_{\lambda_0}}.$$

So, we have

$$|y_n| \leqslant \mu_{\lambda_0} \| \phi \| \max_{s=-r,\ldots,0} \prod_{k=s}^{-1} h_{\lambda_0}(k) + \frac{1 + \mu_{\lambda_0}}{1 + \gamma_{\lambda_0}} |L_{\lambda_0}(\phi)| \quad \text{for } n \geqslant 0. \tag{4.3}$$

Furthermore, we observe that the equalities (3.1), (3.4) and (3.6) (in the proof of Theorem 1) hold. By using these equalities with $n=0$, from the definition of $L_{\lambda_0}(\phi)$ we get

$$
\begin{aligned}
|L_{\lambda_0}(\phi)| \leqslant {} & |\phi_0| + \sum_{i\in I} |c_i| \left(|\phi_{-\sigma_i}| + \left\{ \sum_{s=-\sigma_i}^{-1} \left| 1 - \frac{1}{h_{\lambda_0}(s)} \right| \left[\prod_{k=s}^{-1} h_{\lambda_0}(k) \right] |\phi_s| \right\} \lambda_0^{-\sigma_i} \right) \\
& + \sum_{j\in J} \left\{ \sum_{s=-\tau_j}^{-1} \left[\prod_{k=s+1}^{-1} h_{\lambda_0}(k) \right] |\widetilde{b}_j(s)| |\phi_s| \right\} \lambda_0^{-\tau_j} \\
\leqslant {} & \| \phi \| + \left(\sum_{i\in I} |c_i| \left\{ 1 + \left[\sum_{s=-\sigma_i}^{-1} \left| 1 - \frac{1}{h_{\lambda_0}(s)} \right| \prod_{k=s}^{-1} h_{\lambda_0}(k) \right] \lambda_0^{-\sigma_i} \right\} \right. \\
& \left. + \sum_{j\in J} \left\{ \sum_{s=-\tau_j}^{-1} \left[\prod_{k=s+1}^{-1} h_{\lambda_0}(k) \right] |\widetilde{b}_j(s)| \right\} \lambda_0^{-\tau_j} \right) \| \phi \| \\
= {} & \| \phi \| + \left(\sum_{i\in I} |c_i| \left[\prod_{k=-\sigma_i}^{-1} h_{\lambda_0}(k) + \sum_{s=-\sigma_i}^{-1} \left| 1 - \frac{1}{h_{\lambda_0}(s)} \right| \prod_{k=s}^{-1} h_{\lambda_0}(k) \right] \lambda_0^{-\sigma_i} \right. \\
& \left. + \sum_{j\in J} \left\{ \sum_{s=-\tau_j}^{-1} \left[\prod_{k=s}^{-1} h_{\lambda_0}(k) \right] \frac{1}{h_{\lambda_0}(s)} |\widetilde{b}_j(s)| \right\} \lambda_0^{-\tau_j} \right) \| \phi \| \\
\leqslant {} & \| \phi \| + \left\{ \sum_{i\in I} |c_i| \left[1 + \sum_{s=-\sigma_i}^{-1} \left| 1 - \frac{1}{h_{\lambda_0}(s)} \right| \right] \lambda_0^{-\sigma_i} \right. \\
& \left. + \sum_{j\in J} \left[\sum_{s=-\tau_j}^{-1} \frac{1}{h_{\lambda_0}(s)} |\widetilde{b}_j(s)| \right] \lambda_0^{-\tau_j} \right\} \left[\max_{s=-r,\ldots,0} \prod_{k=s}^{-1} h_{\lambda_0}(k) \right] \| \phi \| \\
= {} & \| \phi \| + \left\{ \sum_{i\in I} |c_i| \left[1 + \sum_{s=0}^{\sigma_i-1} \left| 1 - \frac{1}{h_{\lambda_0}(s)} \right| \right] \lambda_0^{-\sigma_i} \right. \\
& \left. + \sum_{j\in J} \left[\sum_{s=0}^{\tau_j-1} \frac{1}{h_{\lambda_0}(s)} |b_j(s)| \right] \lambda_0^{-\tau_j} \right\} \left[\max_{s=-r,\ldots,0} \prod_{k=s}^{-1} h_{\lambda_0}(k) \right] \| \phi \| .
\end{aligned}
$$

Thus, by the definition of μ_{λ_0},

$$
|L_{\lambda_0}(\phi)| \leqslant \left[1 + \mu_{\lambda_0} \max_{s=-r,\ldots,0} \prod_{k=s}^{-1} h_{\lambda_0}(k) \right] \| \phi \| .
$$

In view of the last inequality, (4.3) gives

$$
|y_n| \leqslant \left[\frac{1+\mu_{\lambda_0}}{1+\gamma_{\lambda_0}} + \mu_{\lambda_0} \left(1 + \frac{1+\mu_{\lambda_0}}{1+\gamma_{\lambda_0}} \right) \max_{s=-r,\ldots,0} \prod_{k=s}^{-1} h_{\lambda_0}(k) \right] \| \phi \| \quad \text{for } n \geqslant 0.
$$

Hence, by the definitions of N_{λ_0} and $(y_n)_{n\geqslant -r}$, we have

$$|x_n| \leqslant N_{\lambda_0} \parallel \phi \parallel H_{\lambda_0}(n) \quad \text{for all } n \geqslant 0. \tag{4.4}$$

Inequality (4.4) is the same with that in the conclusion of the first part of the theorem.

It remains to establish the stability criteria (i), (ii) and (iii) of the theorem. Making use of (4.4) and following the same lines as in the proof of Corollary 3 in Kordonis et al. [14], we can prove (i) and (ii). We will proceed to the proof of (iii). Let $(\mathrm{G}_3(\lambda_0))$ be satisfied and assume, for the sake of contradiction, that the trivial solution of (E) is stable (at 0). Then there exists $\delta \equiv \delta(1) > 0$ such that, for any $\phi = (\phi_n)_{n=-r}^{0}$ in Φ with $\parallel \phi \parallel < \delta$, the solution $(x_n)_{n\geqslant -r}$ of (E)–(C) satisfies

$$|x_n| < 1 \quad \text{for all } n \geqslant -r. \tag{4.5}$$

Set

$$\phi_n^0 = \left[\prod_{k=n}^{-1} h_{\lambda_0}(k)\right]^{-1} \quad \text{for } n = -r, \ldots, 0.$$

Then $\phi^0 = (\phi_n^0)_{n=-r}^{0}$ belongs to Φ and, as we have seen in Section 2, it holds

$$L_{\lambda_0}(\phi^0) = 1 + \gamma_{\lambda_0} > 0.$$

Furthermore, let δ_1 be a number with $0 < \delta_1 < \delta$ and set

$$\phi_n = \frac{\delta_1}{\parallel \phi^0 \parallel}\, \phi_n^0 \quad \text{for } n = -r, \ldots, 0.$$

Clearly, $\phi = (\phi_n)_{n=-r}^{0}$ is an element of Φ such that $\parallel \phi \parallel = \delta_1 < \delta$. For this ϕ, the solution $(x_n)_{n\geqslant -r}$ of (E)–(C) satisfies (4.5). Hence, the sequence $(x_n)_{n\geqslant -r}$ is always bounded, which, because of $(\mathrm{G}_3(\lambda_0))$, guarantees that

$$\liminf_{n\to\infty} \frac{|x_n|}{H_{\lambda_0}(n)} = 0.$$

On the other hand, by Theorem 1, we have

$$\lim_{n\to\infty} \frac{x_n}{H_{\lambda_0}(n)} = \frac{L_{\lambda_0}(\phi)}{1+\gamma_{\lambda_0}} = \frac{(\delta_1/ \parallel \phi^0 \parallel) L_{\lambda_0}(\phi^0)}{1+\gamma_{\lambda_0}} = \frac{\delta_1}{\parallel \phi^0 \parallel} > 0.$$

We have arrived at a contradiction, which shows that the trivial solution of (E) is unstable (at 0).

The proof of the theorem is now complete.

5. The special cases of periodic nonneutral delay difference equations and of autonomous neutral delay difference equations

Let us consider the special case of (nonneutral) delay difference equations with periodic coefficients and constant delays, where the coefficients have a common period and the delays are multiples of this period. More precisely, let us consider the (nonneutral) delay difference equation

$$\Delta x_n = a(n)x_n + \sum_{j\in J} b_j(n)x_{n-\tau_j}. \tag{E$_0$}$$

Eq. (E$_0$) can be obtained (as a special case) from (E) *by taking* $c_i = 0$ for $i \in I$ and considering the initial segment of natural numbers I and the delays σ_i for $i \in I$ to be chosen arbitrarily so that: σ_i for $i \in I$ are positive integers such that $\sigma_{i_1} \neq \sigma_{i_2}$ for $i_1, i_2 \in I$ with $i_1 \neq i_2$; there exist positive integers ℓ_i for $i \in I$ such that $\sigma_i = \ell_i T$ for $i \in I$; and $\sigma \leqslant \tau$. (For example, it can be considered that $I = J$, and $\sigma_i = \tau_i$ for $i \in I$.)

As it concerns the (nonneutral) delay difference equation (E$_0$), we have the integer τ instead of r and the set

$$\Phi_0 = \{\phi = (\phi_n)_{n=-\tau}^{0} : \ \phi_n \in \mathbf{R} \quad \text{for } n = -\tau, \ldots, 0\}$$

in place of Φ. By a solution of (E$_0$) we mean a sequence $(x_n)_{n\geqslant -\tau}$ of real numbers, which satisfies (E$_0$) for $n \geqslant 0$. For the (nonneutral) difference equation (E$_0$), the initial condition (C) becomes

$$x_n = \phi_n \quad \text{for } n = -\tau, \ldots, 0. \tag{C$_0$}$$

The so-called *characteristic equation* of the (nonneutral) delay difference equation (E$_0$) is

$$\lambda^T = \prod_{k=0}^{T-1} \left[1 + a(k) + \sum_{j\in J} b_j(k)\lambda^{-\tau_j} \right]. \tag{*$_0$}$$

Let λ_0 be a positive root of the characteristic equation (*)$_0$. Clearly, (p$_0(\lambda_0)$) holds by itself, and

$$h_{\lambda_0}(n) = 1 + \widetilde{a}(n) + \sum_{j\in J} \widetilde{b}_j(n)\lambda_0^{-\tau_j} \quad \text{for } n \geqslant -\tau.$$

Also, (p(λ_0)) can be written as follows:

(p$^0(\lambda_0)$) If $T > 1$, then

$$h_{\lambda_0}(k) \equiv 1 + a(k) + \sum_{j\in J} b_j(k)\lambda_0^{-\tau_j} > 0 \quad (k = 1, \ldots, T-1).$$

Moreover, when λ_0 has the property (p$^0(\lambda_0)$), the property (P(λ_0)) reduces to

$$\sum_{j\in J} \left[\sum_{s=0}^{\tau_j - 1} \frac{1}{h_{\lambda_0}(s)} |b_j(s)| \right] \lambda_0^{-\tau_j} < 1. \tag{P$^0(\lambda_0)$}$$

Consider a positive root λ_0 of the characteristic equation (*)$_0$ with the properties (p$^0(\lambda_0)$) and (P$^0(\lambda_0)$). The constants γ_{λ_0} and $L_{\lambda_0}(\phi)$ (where $\phi = (\phi_n)_{n=-\tau}^{0}$ in Φ_0) defined in Theorem 1 take respectively the forms

$$\gamma_{\lambda_0} = \sum_{j\in J} \left[\sum_{s=0}^{\tau_j - 1} \frac{1}{h_{\lambda_0}(s)} b_j(s) \right] \lambda_0^{-\tau_j}$$

and

$$L_{\lambda_0}(\phi) = \phi_0 + \sum_{j \in J} \left\{ \sum_{s=-\tau_j}^{-1} \left[\prod_{k=s+1}^{-1} h_{\lambda_0}(k) \right] \widetilde{b}_j(s)\phi_s \right\} \lambda_0^{-\tau_j}.$$

Also, the numbers μ_{λ_0} and N_{λ_0} considered in Theorem 2 are given here by

$$\mu_{\lambda_0} = \sum_{j \in J} \left[\sum_{s=0}^{\tau_j - 1} \frac{1}{h_{\lambda_0}(s)} |b_j(s)| \right] \lambda_0^{-\tau_j}$$

and

$$N_{\lambda_0} = \frac{1+\mu_{\lambda_0}}{1+\gamma_{\lambda_0}} + \mu_{\lambda_0} \left(1 + \frac{1+\mu_{\lambda_0}}{1+\gamma_{\lambda_0}} \right) \max_{s=-\tau,\dots,0} \prod_{k=s}^{-1} h_{\lambda_0}(k).$$

Moreover, condition (Q_1) remains the same and condition (Q_2) becomes

$$\sum_{j \in J} \left[\sum_{s=0}^{\tau_j - 1} |b_j(s)| \right] < 1. \tag{Q_2^0}$$

In view of the above observations, we can easily see that an application of our main results to the special case of the (nonneutral) delay difference equation (E_0) leads to the main results of the paper by Kordonis et al. [14].

Now, let us especially consider the autonomous case, i.e. the special case of the autonomous neutral delay difference equation

$$\Delta \left(x_n + \sum_{i \in I} c_i x_{n-\sigma_i} \right) = a x_n + \sum_{j \in J} b_j x_{n-\tau_j}, \tag{E$'$}$$

where *I and J are initial segments of natural numbers, c_i for $i \in I$, a and $b_j \neq 0$ for $j \in J$ are real numbers, and σ_i for $i \in I$ and τ_j for $j \in J$ are positive integers such that $\sigma_{i_1} \neq \sigma_{i_2}$ $(i_1, i_2 \in I;\ i_1 \neq i_2)$ and $\tau_{j_1} \neq \tau_{j_2}$ $(j_1, j_2 \in J;\ j_1 \neq j_2)$.*

The constant coefficients a and b_j for $j \in J$ of the difference equation (E$'$) can be considered as T-periodic sequences of real numbers with $T = 1$. The assumption, that there exist positive integers ℓ_i for $i \in I$ and m_j for $j \in J$ such that $\sigma_i = \ell_i T$ for $i \in I$ and $\tau_j = m_j T$ for $j \in J$, holds by itself. So, (E$'$) can be obtained (as a special case) from (E).

The characteristic equation of the autonomous difference equation (E$'$) is

$$(\lambda - 1) \left(1 + \sum_{i \in I} c_i \lambda^{-\sigma_i} \right) = a + \sum_{j \in J} b_j \lambda^{-\tau_j}. \tag{*$'$}$$

(Clearly, $(*)'$ can be obtained from (*) for $T = 1$.)

Note that, as it is well-known, the trivial solution of the *autonomous* neutral delay difference equation (E′) is *uniformly stable* (respectively, *uniformly asymptotically stable*) if and only if it is stable (at 0) (respectively, asymptotically stable (at 0)).

Consider a positive root λ_0 of the characteristic equation $(^*)'$. If λ_0 has the property ($\mathrm{p}_0(\lambda_0)$), then

$$h_{\lambda_0}(n) = 1 + \frac{1}{1+\sum_{i\in I} c_i \lambda_0^{-\sigma_i}} \left(a + \sum_{j\in J} b_j \lambda_0^{-\tau_j} \right) = \lambda_0 \quad \text{for } n \geqslant -r$$

and ($\mathrm{p}(\lambda_0)$) holds by itself, while ($\mathrm{P}(\lambda_0)$) becomes

$$\sum_{i\in I} |c_i| \left(1 + \left| 1 - \frac{1}{\lambda_0} \right| \sigma_i \right) \lambda_0^{-\sigma_i} + \frac{1}{\lambda_0} \sum_{j\in J} |b_j| \tau_j \lambda_0^{-\tau_j} < 1. \tag{$\mathrm{P}'(\lambda_0)$}$$

Note that ($\mathrm{P}'(\lambda_0)$) can be defined without the restriction that λ_0 has the property ($\mathrm{p}_0(\lambda_0)$). On the other hand, it follows from ($\mathrm{P}'(\lambda_0)$) that

$$\sum_{i\in I} |c_i| \lambda_0^{-\sigma_i} < 1,$$

which implies that

$$1 + \sum_{i\in I} c_i \lambda_0^{-\sigma_i} > 0.$$

That is, if λ_0 has the property ($\mathrm{P}'(\lambda_0)$), then it also has the property ($\mathrm{p}_0(\lambda_0)$). So, the only property needed for the positive root λ_0 of $(^*)'$ is the property ($\mathrm{P}'(\lambda_0)$). Furthermore, we have

$$H_{\lambda_0}(n) = \lambda_0^n \quad \text{for } n \geqslant -r.$$

Next, let λ_0 be a positive root of the characteristic equation $(^*)'$ with the property ($\mathrm{P}'(\lambda_0)$). The numbers γ_{λ_0} and $L_{\lambda_0}(\phi)$ (where $\phi = (\phi_n)_{n=-r}^0$ in Φ) defined in Theorem 1 are written in the forms

$$\gamma_{\lambda_0} = \sum_{i\in I} c_i \left[1 - \left(1 - \frac{1}{\lambda_0} \right) \sigma_i \right] \lambda_0^{-\sigma_i} + \frac{1}{\lambda_0} \sum_{j\in J} b_j \tau_j \lambda_0^{-\tau_j}$$

and

$$\begin{aligned} L_{\lambda_0}(\phi) = \phi_0 &+ \sum_{i\in I} c_i \left[\phi_{-\sigma_i} - \left(1 - \frac{1}{\lambda_0} \right) \left(\sum_{s=-\sigma_i}^{-1} \lambda_0^{-s} \phi_s \right) \lambda_0^{-\sigma_i} \right] \\ &+ \frac{1}{\lambda_0} \sum_{j\in J} b_j \left(\sum_{s=-\tau_j}^{-1} \lambda_0^{-s} \phi_s \right) \lambda_0^{-\tau_j} \end{aligned}$$

respectively. Also, the constants μ_{λ_0} and N_{λ_0} considered in Theorem 2 take respectively the forms

$$\mu_{\lambda_0} = \sum_{i\in I} |c_i| \left(1 + \left| 1 - \frac{1}{\lambda_0} \right| \sigma_i \right) \lambda_0^{-\sigma_i} + \frac{1}{\lambda_0} \sum_{j\in J} |b_j| \tau_j \lambda_0^{-\tau_j}$$

and

$$N_{\lambda_0} = \frac{1+\mu_{\lambda_0}}{1+\gamma_{\lambda_0}} + \mu_{\lambda_0}\left(1+\frac{1+\mu_{\lambda_0}}{1+\gamma_{\lambda_0}}\right)\max\{1, \lambda_0^r\}.$$

Moreover, we can immediately see that each one of $(G_1(\lambda_0))$, $(G_2(\lambda_0))$ or $(G_3(\lambda_0))$ holds, if and only if, the positive root λ_0 of $(^*)'$ satisfies $\lambda_0 \leqslant 1$, $\lambda_0 < 1$ or $\lambda_0 > 1$ respectively.

Finally, we notice that (Q_1) and (Q_2) become, respectively,

$$a + \sum_{j\in J} b_j = 0 \tag{Q_1'}$$

and

$$\sum_{i\in I} |c_i| + \sum_{j\in J} |b_j|\tau_j < 1. \tag{Q_2'}$$

By taking into account the above facts, we can apply our main results to the special case of the autonomous neutral delay difference equation (E'); such an application leads to the main results in the paper by Kordonis and Philos [12]. It must be noted that sufficient conditions for the characteristic equation $(^*)'$ of the autonomous neutral delay difference equation (E') to have a positive root λ_0 with the property $(P'(\lambda_0))$ have been given in [12].

Before closing this section and ending the paper, let us consider the simple case of the autonomous (nonneutral) delay difference equation

$$\Delta x_n = a x_n + \sum_{j\in J} b_j x_{n-\tau_j}, \tag{E_0'}$$

which can be obtained (as a special case) either from (E_0) or from (E'). The characteristic equation of (E_0') is

$$\lambda - 1 = a + \sum_{j\in J} b_j \lambda^{-\tau_j}. \tag{$(^*)_0'$}$$

Our results are applicable to the most simple case of (E_0'). For the results obtained by such an application, see [14, Section 6]. Finally, we notice that some conditions, under which the characteristic equation $(^*)_0'$ has a positive root λ_0 with the property

$$\frac{1}{\lambda_0}\sum_{j\in J} |b_j|\tau_j \lambda_0^{-\tau_j} < 1,$$

have been obtained in [14].

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